Indian Statistical Institute
Second Semester Examination 2003-2004
B.Math(Hons.) III Year
Optimization

Time: 3 hrs Date:06-05-04 Max. Marks : 75

Answer all Questions

1. Prove that exactly one of the following two systems has a solution

   System I: $Ax \geq 0, \quad x \geq 0, \quad c_x^t > 0$

   System II: $A^t y \geq c$ and $y \leq 0$.

   [You may use Farka's theorem) [10]

2. Let $\alpha, \beta, \in R$ consider the LP-model:
   $Minimize(x_1 + x_s)$
   such that
   $\alpha x_1 + \beta x_2 \geq 1$
   $x_1 \geq 0, \quad x_2$ -unrestricted.

   Determine necessary and sufficient conditions for α and β such that the model:
   a) is infeasible
   b) is feasible
   c) has an optimal solution
   d) is feasible and the optimal solution is unbounded.
   e) has a multiple optimal solution. [10]

3. Consider the Linear Program

$$Minimize \sum_{j=1}^{n} c_j x_j = Z$$

   .

   subject to

$$\sum_{j=1}^{n} P_j x_j = b, \qquad P_j \in R^m$$

$$x_j \geq 0 \text{ for } \quad j = 1, 2 \ldots n$$

   Suppose that $Z = Z^*, \quad x_j = x^*$ for $j = 1, 2 \ldots n$ is a nondegenerate optimal basic feasible solution. A new activity vector is being considered for augmenting the problem to [i.e., a new variable is added and the problem is changed to]

Minimize $\sum_{j=1}^{n} c_j x_j + c_{n+1}, x_{n+1} = Z$

subject to $\sum_{j=1}^{n} P_j x_j + P_{n+1} x_{n+1} = b, \quad P_j \in R^m, \quad x_j \geq 0$

a) How would you test whether the column $(n+1)$ is a candidate for improving the solution?

b) Assume that the column $n+1$ passes the test and x_{n+1} enters the basic set displacing x_r. Let $Z = \hat{Z}$ and $x_j = \hat{x}_j$ for $j = 1, 2 \ldots n+1$ be the updated feasible solution. Prove that it is a strict improvement, that is $\hat{Z} < Z^*$.

c) Suppose the augmented problem is now iterated to a new optimal solution. Prove that X_{n+1} once in never leaves the basic set. [10]

4. a) If the primal system $\min Z = c^t x, \quad Ax \geq b, \ x \geq 0$ has a feasible solution and the dual system $\max V = b^t y, \quad A^T y \leq c, \quad y \geq 0$ has a feasible solution then there exist optimal feasible solutions.
$x = x^*$ and $y = y^*$ to the primal and dual systems such that
$b^t y^* = maxV = \min Z = c^t x^*$

b) Use the dual Simplex method to solve
Minimize $x_1 + 2x_2$
subject to

$$\begin{aligned} 2x_1 - 4x_2 &\geq 2 \\ 2x_1 - 2x_2 &\geq 7 \\ x_1 + 3x_2 &\geq -2 \end{aligned}$$

$x_1 \geq 0, x_2 \geq 0$ [10]

5. Find the optimal solution for the following Transportation problem [10]

	R_1	R_2	R_3	R_4	Availability
W_1	10	5	6	7	25
W_2	8	2	7	6	25
W_3	9	3	4	8	50
Requirements	15	20	30	35	

6. Apply Dikin's affine scaling procedure with respect to the interior point $\bar{x}$ on the following LP Model; show that a unit ball (circle in R^2, line segment in R) with centre 1 and radius 1 is completely contained in the "transformed feasible region".

$\max x_1 + 2x_2$

subject to $2x_1 + x_2 + x_3 = 2, \quad x_1, x_2, x_3 \geq 0$

$\bar{x} = [0.1, 0.1, 1.6]^t$ [10]

7. Prove that is both the Primal feasible region F_p and the dual feasible region F_D are nonempty then at least one of the corresponding LP-models (P) or (D) has an unbounded feasible solution. [10]

Solution: $[F_P = \{x \in R^n | Ax = b, x \geq 0\}; F_D = \{y \in R^m | A^t y \leq c\}]$

8. a) There are five pumps available for developing five wells. The efficiency of each pump in producing the maximum yield at each well is given below

		Well			
Pump	1	2	3	4	5
1	45	40	65	30	55
2	50	30	25	60	30
3	25	20	15	20	40
4	35	25	30	25	20
5	80	60	60	70	50

In what way should the pump be assigned so as to maximize the overall efficiency !

b) Show that all bases for the transportation problem are triangular [10]